RECURRENCE IN GENERIC STAIRCASES

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Abstract. The straight-line flow on almost every staircase and on almost every square tiled staircase is recurrent. For almost every square tiled staircase the set of periodic orbits is dense in the phase space.

1. Introduction

A compact translation surface is a surface which can be obtained by edge-to-edge gluing of finitely many polygons in the plane using only translations. Since the seminal work of Veech in 1989 [Ve] the study of compact translation surfaces of finite area have developed extensively. The study of translation surfaces of infinite area, obtained by gluing countably many polygons via translations, has only recently begun. A natural class of infinite translation surfaces, staircases, were introduced in [HuWe] and studied in [HoWe]. Billiards in irrational polygons give rise to another class of infinite translation surfaces [GuTr].

One of the first dynamic properties of infinite translation surfaces one needs to understand is the almost sure recurrence of the straight-line flow. Recurrence of infinite translation surfaces have been investigated in [GuTr], [Ho], [HoWe], [HuWe], [HuLeTr], [ScTr], and [Tr]. Hubert and Weiss studied a special staircase surface, shown in Figure 2 on the left [HuWe]. They showed that the straight-line flow is almost surely recurrent and completely classified the ergodic measures as well as the periodic points. In [HoWe], Hooper and Weiss classified the periodic square tiled staircases which are almost surely recurrent.

In this article we study non-periodic staircases. We show that almost all staircases are recurrent. Two different notions of almost every staircase will be given, one for square tiled staircases, the other more general. We also show that the square tiled ones have dense set of directions for which all regular orbits are periodic. These results follow from approximating arbitrary staircases by periodic ones.

2. Arbitrary staircases

Consider $\Sigma' := (\mathbb{R}_+ \times \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+)^{\mathbb{Z}}$ with the product topology (here $\mathbb{R}_+ := \{x : x \geq 0\}$ and $\mathbb{R}_+^* := R_+ \setminus \{0\}$). Note that Σ' is not compact. Fix $v := (l, h, l', h') \in \Sigma'$. The staircase T_v will be formed as follows (see Figure 1). All rectangles are oriented to have horizontal and vertical sides. There are four types of rectangles: the

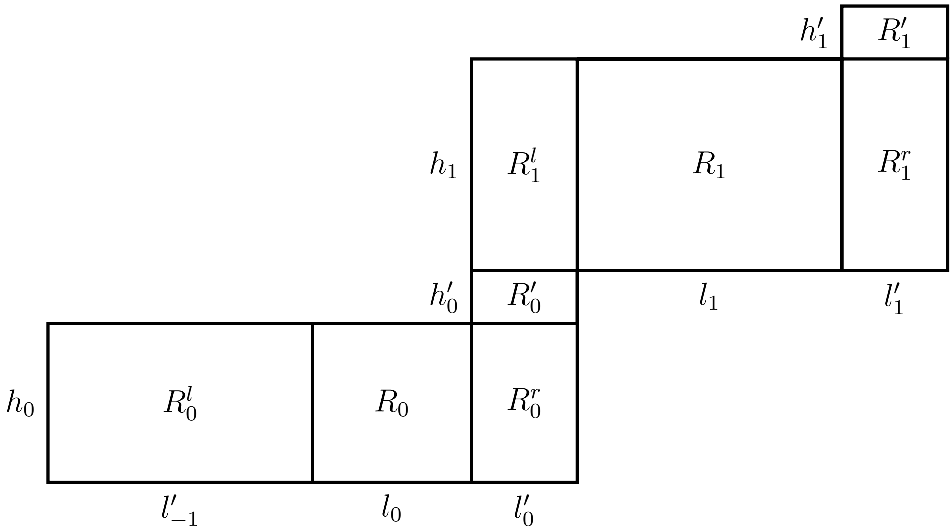

FIGURE 1. An arbitrary staircase

rectangle R_n has length l_n and height h_n, R'_n has dimensions l'_n, h'_n, R^l_n has dimensions l'_n, h_n, and R^r_n has dimensions l'_{n-1}, h_n. The staircase consists of, for each $n \in \mathbb{Z}$, one copy of each of these rectangles. We place the rectangle R^r_n to the right of rectangle R_n and we place R^l_n to the left of R_n. Above R^r_n we place R'_n while we place R'_{n-1} below R^l_n. Continue this procedure inductively. Touching edges and edges which are a horizontal or vertical translation of one another are identified.

We consider the straight-line flow ψ_t on T_v in a fixed direction $\theta \in \mathbb{S}^1$. The phase volume is the natural invariant measure of this flow. A staircase is called *recurrent in the direction* θ if for any set of positive measure in the phase space a.e. orbit returns to this set and *recurrent* if it is recurrent in almost every direction.

Consider the shift transformation $\sigma : \Sigma' \circlearrowleft$ defined by $\sigma(v)_n = v_{n+1}$. Consider any σ-invariant ergodic probability measure ν on Σ' whose support contains at least one periodic staircase. For example, let $f : \mathbb{R}_+ \circlearrowleft$ be a continuous Lebesgue integrable function with integral 1 and let λ be the probability measure on $\mathbb{R}_+$ whose Radon Nykodym derivative with respect to the length measure is f. Then $\lambda^{(4)} := \lambda \times \lambda \times \lambda \times \lambda$ is a probability measure on $\mathbb{R}_+ \times \mathbb{R}^*_+ \times \mathbb{R}^*_+ \times \mathbb{R}_+$, and thus the product measure ν_f determined by $\lambda^{(4)}$ is a natural shift invariant probability measure on staircases. Since this measure is Bernoulli, periodic points are dense in its support.

Theorem 1. *For ν a.e. $v \in \Sigma'$ the staircase T_v is recurrent.*

Remarque: we will show the the recurrent directions of the periodic staircase in the support of ν are recurrent directions for T_v.

3. Square tiles staircases

We consider a special case of the above construction with a different coding. Let $\Sigma_2 := \{0,1\}^{\mathbb{Z}}$ and endow Σ_2 with the product topology. For each $s \in \Sigma_2$ we construct the flat surface T_s as follows (see Figure 2). The surface T_s consists of an infinite number of unit squares labelled by $\mathbb{Z}$. The sides of the squares are parallel to the x and y axes. The $n+1$st square is placed above the nth square if $s_n = 0$ and otherwise it is to the right of the nth square. The surface T_s is obtained by identifying the common edges and identifying pairs of edges which are a horizontal or vertical translation of one another. For each $s \in \Sigma_2$ other than those ending with an infinite sequence of 0's or 1's (in the positive or negative direction) we can find a unique $v(s) \in \Sigma'$ which describes the same staircase with with $h_n = l'_n = 1$ and $h'_n \in \mathbb{N}, l_n \in \mathbb{N}$ for all n. We leave the exact computation to the reader.

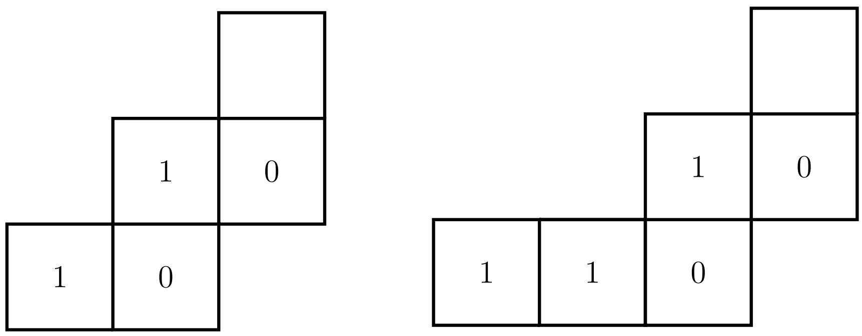

Figure 2. The staircases $\cdot\cdot\,1010\,\cdot\cdot$ and $\cdot\cdot\,11010\,\cdot\cdot$

We consider the shift transformation σ on Σ_2. Consider any σ-invariant ergodic probability measure μ on Σ whose support contains a periodic staircase other than 0^∞ or 1^∞. Any measure with full support, in particular the Bernoulli measures, satisfy this assumption.

Theorem 2. *For μ a.e. $s \in \Sigma_2$ the staircase T_s is recurrent.*

A direction $\theta \in \mathbb{S}^1$ is called *purely periodic* if all regular orbits in this direction are periodic. We additionally assume that the measure μ includes the periodic point $(10)^\infty$ (satisfied by Bernoulli measures).

Theorem 3. *For μ a.e. $s \in \Sigma_2$, the staircase T_s has a dense set of purely periodic directions.*

Remark: we will show that the set of purely periodic directions of T_s includes the set of purely periodic directions of $T_{(10)^\infty}$ which was shown to be dense by [HuWe].

4. Proofs

The ω limit set of $v \in \Sigma'$ is $\cap_{n\geq 0}\overline{\{\sigma^k v : k > n\}}$ and the α limit set is defined similarly with σ^k replaced by σ^{-k}. Theorems 1 and 2 will follow almost immediately from the following result.

Theorem 4. *If there is a periodic point v^+ in the intersection of the α and ω limit sets of $v \in \Sigma'$, then the staircase T_v is recurrent for any direction θ for which T_{v^+} is recurrent.*

Proof of Theorem 1. By standard arguments on skew products due to Schmidt ([Sc], Theorem 11.4) (see also Proposition 10 of [HoWe]) which we will sketch below, the staircase T_{v^+} is recurrent in a.e. direction θ , and thus Theorem 1 follows from Theorem 4.

Suppose (X, μ) is a finite measure space and $R : X \to X$ is a measurable transformation preserving μ which is ergodic. For a measurable $f : X \to \mathbb{Z}$, $f \in L^1(X, \mu)$ define $X_f := X \times \mathbb{Z}$ and

$$R_f : X_f \to X_f, \quad R_f(x, k) = (Rx, k + f(x)).$$

Schmidt showed that R_f is recurrent if and only if $\int f\, d\mu = 0$. We apply this result to a periodic staircase in the following way. Consider a finite crossection, say the side of one of our rectangles. If we quotient the staircase T_{v^+} by its period, then the first return map (for a fixed direction) to this crosssection is an interval exchange map R, and Kerckhoff, Masur and Smillie have shown that this interval exchange map is ergodic for almost every direction [KeMaSm]. If we restrict to an ergodic direction, and consider the lifts of this cross section to T_{v^+} we get a map of the form R_f. From our staircase construction we have $\int f\, d\mu = 0$ (this is holonomy 0 in the language of [HoWe]), thus we can apply Schmidt's theorem to conclude the recurrence of periodic staircases. □

Proof of Theorem 2. The ω (resp. α) limit set of $s \in \Sigma_2$ includes a periodic point other that 0^∞ and 1^∞ if and only if the ω (resp. α) limit set of the corresponding $v(s) \in \Sigma'$ contains a periodic point. Thus we can conclude the proof by applying Theorem 4. □

Proof of Theorem 4. Let $T = T_v$ and $T^+ = T_{v^+}$. Fix a recurrent direction θ for T_{v^+}. For each $n \in \mathbb{Z}$, let L_n denote the left boundary of the rectangle $R_n^r \cup R_n' \cup R_{n+1}^l$. Let $\mathcal{L}_n = L_n \times \theta$ and let ϕ be the first return map of the straight-line flow on T to the set $\mathcal{L} := \cup_{n\in\mathbb{Z}}\mathcal{L}_n$. Denote corresponding objects in T^+ with a superscript $+$, for example $\mathcal{L}^+$ and ϕ^+. By assumption the map ϕ^+ is recurrent.

The idea of the proof is quite simple. First we make quantitative estimates on the recurrence of ϕ^+ on the periodic table T^+. There are arbitrarily large finite parts of T which are ε close to a part of T^+, thus we can transfer these estimates to T. Care must be taken for orbits which come to close to a corner, and thus can hit a different sequence of sides in T and T^+. Conceptually the transfer should be clear, but the formal proof is a bit technical since the measure spaces are not identical, but only close. Since the map is invertible this will imply its recurrence, and finally we give an argument to conclude the recurrence of the flow.

Fix $\varepsilon > 0$. Fix $N > 0$ so that the ϕ^+ orbit of at least $(1-\varepsilon)\%$ of the points in $\mathcal{L}_0^+$ recurs to $\mathcal{L}_0^+$ before a time N. Consider $D_N^+ := \cup_{-N\le n\le N} L_n^+$ and $\mathcal{D}_N^+ = D_N^+ \times \theta$. There are 4 corners of squares which are in the set L_n^+. After identification, all corners of the table T^+ belong to the set L^+. In total, there are $4(2N+1)$ corners in D_N^+ (without identifications).

Take a small $\varepsilon' = \varepsilon(h_0 + h_0' + h_1)/(N \cdot 4(2N+1))$ neighborhood $B_{\varepsilon'}$ of these corners in D_N^+. This set has measure (length) at most $\varepsilon(h_0+h_0'+h_1)/N$. Let $\mathcal{B}_{\varepsilon'} := B_{\varepsilon'} \times \theta$. Let $C := \mathcal{L}_0 \cap \cup_{i=0}^{N-1} (\phi^+)^{-i}\mathcal{B}_{\varepsilon'}$, since ϕ^+ is measure preserving, the measure of C is at most $N \times \varepsilon(h_0 + h_0'+h_1)/N = \varepsilon(h_0+h_0'+h_1)$. Since the total measure of $\mathcal{L}_0$ is $h_0+h_0'+h_1$ we conclude that $(1-\varepsilon)\%$ of the points in $\mathcal{L}_0$ are not in C, i.e. the first N images of this $(1-\varepsilon)\%$ set of points stay at least a distance ε' away from the singular points.

Combining these two facts yields a set $G^+ \subset \mathcal{L}_0$ consisting of at least $(1-2\varepsilon)\%$ of the points in $\mathcal{L}_0^+$, such that the points of G^+ recur before time N without having visited an ε' neighborhood of a singular point.

Now consider the staircase T. Let $H_n^+ := h_n^+ + {h'}_n^+ + h_{n+1}^+$ be the length of L_n^+ (denote by H_n the corresponding length in T) and $H^+ = \inf_{n\in\mathbb{Z}} H_n$. Note that since T^+ is periodic H^+ is strictly positive. Since v^+ is a ω limit point of v, for each $\varepsilon' > 0$ and $N > 0$ there is a positive M so that $\max(|l_{i+M} - l_i^+|, |h_{i+M} - h_i^+||, l'_{i+M} - {l'}_i^+|, |h'_{i+M} - {h'}_i^+|) < \varepsilon' H^+/3N^2$ for all $|i| \le N$. Thus, for each $|i| \le N$ we have $|H_i^+ - H_{M+i}| \le \varepsilon' H^+/N^2$. For concreteness, fix the bottom point of each interval L_{M+i} and L_i^+ as its origin. Formally, this yields an identification of $((H_{M+i} - \varepsilon' H^+/N)/H_{M+i})\% = (1 - \varepsilon' H^+/NH_{M+i})\%$ of L_{M+i} with L_i^+ via the "identity map" for each fixed $|i| \le N$. In $\mathbb{R}^2$ this identification moves points by at most a distance of $N(\varepsilon' H^+/N^2) = \varepsilon' H^+/N$.

Consider a point $s^+ \in L_0^+$, $s = id(s^+) \in L_M$ for which the identification exists. Consider the ϕ and ϕ^+ orbits of (s,θ). Let $(s_i,\theta_i) := \phi^i(s,\theta)$ and $(s_i^+,\theta_i^+) := (\phi^+)^i(s,\theta)$. We compare these orbits for times $i \in \{0,1,\dots,N\}$. As long as these orbits visit the "same" sequence of sides they remain parallel, i.e. $\theta_i^+ = \theta_i$ and the orbits diverge at most linearly i.e. the distance $d(s_i^+, s_i) \le i(\varepsilon' H/N)$. If $0 \le i \le N$ then $d(s_i^+, s_i) \le \varepsilon' H$.

In particular, if the point $(s^+,\theta) \in G^+$ then for $0 \le i \le N$ the s_i are a distance at least ε' from the corners, so since $d(s_i, s_i) < \varepsilon'$ they are on the same side and we can apply this observation to conclude that since (s^+,θ) is ϕ^+-recurrent, the point (s,θ) is ϕ-recurrent. This holds for at least $(1-3\varepsilon)\%$ of the points in $\mathcal{L}_M$

Consider $B := \{(s,\theta) \in \mathcal{L}_0 : s_i \notin L_0 \text{ for all } i > 0\}$, i.e. the set of points in $\mathcal{L}_0$ which do not recur to $\mathcal{L}_0$. Since ϕ is invertible, we have $\phi^i B \cap \phi^j B = \emptyset$ for all $i > j$ (otherwise $\phi^{i-j} B \cap B \neq \emptyset$, which contradicts the definition of B). Fix $n > 0$. Thus a.e. point of B can hit each level $\mathcal{L}_n$ $(n \geq 0)$ only a finite number of times. Let $m_n(s,\theta)$ be the last time the orbit of (s,θ) visits $\mathcal{L}_n$, i.e. $s_{m_n} \in L_n$ and $s_j \notin L_n$ for all $j > m_n$. Then $\Phi(s,\theta) := \phi^{m_n(s,\theta)}(s,\theta)$ is a measure preserving map $\Phi : B \to \mathcal{L}_n$ defined almost surely. By definition the image $\Phi(B)$ consists of nonrecurrent points in $\mathcal{L}_n$. In particular if $n = M$ as chosen above then we conclude that the measure of B is at most 3ε. Since $\varepsilon > 0$ was arbitrary, a.e. point in $\mathcal{L}_0$ is ϕ-recurrent. We can repeat this argument with $\mathcal{L}_0$ replaced by $\mathcal{L}_j$ for any fixed j to conclude that a.e. point in $\mathcal{L}_j$ is ϕ-recurrent for all j.

Finally we need to show that the flow ψ_t is recurrent in the direction θ. Consider any small open ball B in the phase space of the flow ψ_t. Flow each non-singular point of $B \times \{\theta\}$ until it hits the set $\mathcal{L}$. Since the ball is open, it has positive phase volume, and its image on the set $\mathcal{L}$ has positive phase area. Almost every of these points is a ϕ-recurrent point by the above.

Fix a nonsingular $x \in B$ such that $(x_N,\theta) := \psi_t(x,\theta) \in \mathcal{L}_N$. Note that by transversality and the Fubini theorem almost every $x \in B$ corresponds to a ϕ-recurrent $(x_N,\theta) \in \mathcal{L}_N$. To conclude the proof we suppose that (x_N,θ) is ϕ-recurrent and we will show that this implies that (x,θ) is ψ_t-recurrent. Choose a open neighborhood U of x small enough that for each $y \in U$ there is a $t(y)$ very close to t such that $\psi_{t(y)}(y,\theta) \in \mathcal{L}_N$. Let $U' = \{\psi_{t(y)}(y,\theta) : y \in U\}$. This is a small neighborhood of $\psi_t(x,\theta)$, and by the above results there is an (arbitrarily large) n such that $\phi^n(x_N,\theta) \in U'$. Thus $\phi^n(x_N,\theta) = \psi_s(x_N,\theta) = \psi_{s+t}(x,\theta) \in U'$ for some large s. Since $\psi_{s+t}(x,\theta)$ is in U' it is the image $\psi_{t(y_0)}(y_0,\theta)$ for some $y_0 \in U$. Thus $\psi_{s+t-t(y_0)}(x,\theta) = (y,\theta)$ with $y \in U$ and we conclude that (x,θ) is ψ recurrent. $\square$

A purely periodic direction is called *strongly parabolic* if the phase space decomposes into an infinite number of cylinders isometric to each other. A periodic staircase is called good if there are a dense set of strongly parabolic directions.

Theorem 5. *If the α and ω limit sets of $s \in \Sigma_2$ include a periodic point $\hat{s} \in \Sigma_2$ such that the staircase $T_{\hat{s}}$ has a dense set of strongly periodic directions, then the staircase T_s has a dense set of purely periodic directions.*

Proof of Theorem 3. The theorem follows from Theorem 5 since Hubert and Weiss have shown that the table $(10)^\infty$ satisfies the assumptions of Theorem 5. □

Proof of Theorem 5. We consider another cross section to the billiard flow, let $\mathcal{D}_n$ be the left side of the nth square of the table T_s and let $\mathcal{D} = \cup_{n \in \mathbb{Z}} \mathcal{D}_n$. Similarly let $\hat{\mathcal{D}}_n$ be the left side of the nth square of $T_{\hat{s}}$. Clearly all non-singular vertical orbits are periodic in T_s. Fix a strongly parabolic direction $\theta \in \mathbb{S}^1$ for $T_{\hat{s}}$ which is not vertical, the flow in the direction θ in T_s is transverse to $\mathcal{D}$. Consider the first return map $\zeta : \mathcal{D} \times \{\theta\} \to \mathcal{D} \times \{\theta\}$ of the flow in this direction. It is of the form

$$\zeta(r, n, \theta) = (r + \alpha \,(\mathrm{mod}\ 1), n + f(s, n, \theta), \theta)$$

for a certain $\alpha = \alpha(\theta)$. In the table T_s one sees arbitrarily large initial pieces of the staircase $T_{\hat{s}}$: for each $N > 0$ there is are $M^- < 0 < M^+$ such that $s_{i+M^\pm} = \hat{s}_i$ for each $i \in \{-N, \ldots, +N\}$. We immediately conclude that since the direction θ is strongly parabolic for $T_{\hat{s}}$ the staircase T_s must have periodic orbits in the direction θ and thus the number α must be rational.

Since θ is strongly parabolic for $T_{\hat{s}}$ there is a t_0 so that all orbits in the direction θ have flow period t_0. Thus we can find an N so that all nonsingular orbits starting in $\hat{\mathcal{D}}_0$ are periodic do not visit any $\hat{\mathcal{D}}_i$ with $|i| > N$. Thus we can conclude that for the table T_s all nonsingular orbits starting the the two squares $T_{M^\pm}$ are periodic.

Now consider any point $x = (r, i, \theta)$ with nonsingular orbit with $M^- < i < M^+$. First of all since $\mathcal{D}_{M^\pm}$ consists completely of periodic orbits the orbit of x can not reach $\mathcal{D}_{M^\pm}$ without being one of these periodic orbits. If the orbit does not reach $\mathcal{D}_{M^\pm}$ then it stays inside the compact region $\cup_{i=M^-+1}^{M^+-1} \mathcal{D}_i$. Since α is rational, and the number of $\mathcal{D}_i$ it visits is finite, the coordinates (r_i, n_i) of the orbit can only take a finite number of values. Thus it must visit some point (r, n) twice. Since the dynamics is invertible the orbit is periodic. The result follows since $M^- \to -\infty$ and $M^+ \to \infty$ as $N \to \infty$ □

5. Remarks

These proofs hold in a more general setting, they use only the one dimensionality of the construction of staircases. For the sake of clarity of exposition we did not try to define a larger class of flat surfaces.

The main idea of proof of this article was developed in a two dimensional setting for the Ehrenfest wind-tree model in [Tr1]. Since the approximation technique is essentially one dimensional in the two dimensional setting it yields only topological results, i.e. a dense G_δ of recurrent wind-tree models with dense purely periodic directions. The recurrence result also holds for a dense G_δ of Lorentz gases [Tr1].

Returning to the one dimensional setting we see that our arguments also yield recurrence for almost every quench Lorentz tube (see [CrLeSe]). The typical quenched Lorentz tubes do not needed to have finite horizon unlike those of [CrLeSe]. Using the hyperbolic structure, it can be shown that these Lorentz tubes are not only recurrent, but also ergodic [LeTr].

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